Frames of Reference and Relative Velocity

Air shows provide elements of both excitement and danger. When high-speed airplanes fly in constant formation (Figure 1), observers on the ground see them moving at high velocity. Seen from the cockpit, however, all the planes appear to have zero velocity. Observers on the ground are in one frame of reference, while the pilots are in the plane’s frame of reference. A frame of reference is a coordinate system relative to which motion is described or observed.

The most common frame of reference that we use as a stationary, or fixed, frame of reference is Earth or the ground. In the examples of motion presented in the previous sections, all objects were assumed to be moving relative to the frame of reference of Earth. Sometimes, however, other frames are chosen for convenience. For example, to analyze the motion of the planets of the solar system, the Sun’s frame of reference is used. If we observe a spot near the rim of a rolling wheel, the wheel or the centre of the wheel is the most convenient frame of reference, as in Figure 2.

\[ \vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE} \]

This equation applies whether the motion is in one, two, or three dimensions. For example, consider the one-dimensional situation in which the wind and the plane are both moving eastward. If the plane’s velocity relative to the air is 430 km/h [E], and the air’s

**Figure 1**
The Canadian Forces Snowbirds fly at velocities of between 400 and 600 km/h (relative to the ground), but when they are flying in formation, as shown here, the velocity of one plane relative to another is zero.

**Figure 2**
(a) The motion of a spot near the rim of a rolling wheel is simple if viewed from the frame of reference of the wheel’s centre.
(b) The motion of the spot is much more complex when viewed from Earth’s frame of reference.

**DID YOU KNOW?**

Viewing the Solar System
It is easy to visualize planets revolving around the Sun, using the Sun as the frame of reference. Ancient astronomers, however, used Earth’s frame of reference to try to explain the observed motion of the planets, but had to invent forces that do not exist. For example, when watching the motion of a planet beyond Earth (such as Mars) against the background of the stars, the planet appears to reverse direction from time to time, much like a flattened “S” pattern. In fact, the planet doesn’t reverse directions; it only appears to do so as Earth, which is closer to the Sun, catches up and then passes the planet.
velocity relative to the ground is 90 km/h [E], then the velocity of the plane relative to the ground is:

\[ \vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE} \]
\[ = 430 \text{ km/h [E]} + 90 \text{ km/h [E]} \]
\[ \vec{v}_{PE} = 520 \text{ km/h [E]} \]

Thus, with a tail wind, the ground speed increases—a logical result. You can easily figure out that the plane’s ground speed in this example would be only 340 km/h [E] if the wind were a head wind (i.e., if \( \vec{v}_{AG} = 90 \text{ km/h [W]} \)).

Before looking at relative velocities in two dimensions, make sure that you understand the pattern of the subscripts used in any relative velocity equation. As shown in Figure 3, the left side of the equation has a single relative velocity, while the right side has the vector addition of two or more relative velocities. Note that the “outside” and the “inside” subscripts on the right side are in the same order as the subscripts on the left side.

\[
\vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE} \quad \vec{v}_{CE} = \vec{v}_{CW} + \vec{v}_{WE} \\
\vec{v}_{LO} = \vec{v}_{LM} + \vec{v}_{MN} + \vec{v}_{NO} \quad \vec{v}_{DG} = \vec{v}_{DE} + \vec{v}_{EF} + \vec{v}_{FG}
\]

**Figure 3**
The pattern in relative velocity equations

### SAMPLE problem 1

An Olympic canoeist, capable of travelling at a speed of 4.5 m/s in still water, is crossing a river that is flowing with a velocity of 3.2 m/s [E]. The river is \( 2.2 \times 10^2 \) m wide.

(a) If the canoe is aimed northward, as in Figure 4, what is its velocity relative to the shore?

(b) How long does the crossing take?

(c) Where is the landing position of the canoe relative to its starting position?

(d) If the canoe landed directly across from the starting position, at what angle would the canoe have been aimed?

![Figure 4](https://via.placeholder.com/150)

**Figure 4**
The situation

**Solution**

Using the subscripts C for the canoe, S for the shore, and W for the water, the known relative velocities are:

\( \vec{v}_{CW} = 4.5 \text{ m/s [N]} \)
\( \vec{v}_{WS} = 3.2 \text{ m/s [E]} \)

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**DID YOU KNOW?**

### Wind Directions

By convention, a west wind is a wind that blows from the west, so its velocity vector points east (e.g., a west wind might be blowing at 45 km/h [E]). A southwest wind has the direction [45° N of E] or [45° E of N].

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**DID YOU KNOW??**

### Navigation Terminology

Air navigators have terms for some of the key concepts of relative velocity. *Airspeed* is the speed of a plane relative to the air. *Wind speed* is the speed of the wind relative to the ground. *Ground speed* is the speed of the plane relative to the ground. The *heading* is the direction in which the plane is aimed. The *course, or track*, is the path relative to Earth or the ground. Marine navigators use “heading,” “course,” and “track” in analogous ways.
(a) Since the unknown is $\vec{v}_{CS}$, we use the relative velocity equation

$$\vec{v}_{CS} = \vec{v}_{CW} + \vec{v}_{WS}$$

$\vec{v}_{CS} = 4.5 \text{ m/s [N]} + 3.2 \text{ m/s [E]}$

Applying the law of Pythagoras, we find:

$$|\vec{v}_{CS}| = \sqrt{(4.5 \text{ m/s})^2 + (3.2 \text{ m/s})^2}$$

$$|\vec{v}_{CS}| = 5.5 \text{ m/s}$$

Trigonometry gives the angle $\theta$ in Figure 4:

$$\theta = \tan^{-1} \frac{3.2 \text{ m/s}}{4.5 \text{ m/s}}$$

$$\theta = 35^\circ$$

The velocity of the canoe relative to the shore is 5.5 m/s [35° E of N].

(b) To determine the time taken to cross the river, we consider only the motion perpendicular to the river.

$$\Delta \vec{d} = 2.2 \times 10^2 \text{ m [N]}$$

$$\vec{v}_{CW} = 4.5 \text{ m/s [N]}$$

$$\Delta t = ?$$

From $\vec{v}_{CW} = \Delta \vec{d} / \Delta t$, we have:

$$\Delta t = \frac{\Delta \vec{d}}{\vec{v}_{CW}}$$

$$= \frac{2.2 \times 10^2 \text{ m [N]}}{4.5 \text{ m/s [N]}}$$

$$\Delta t = 49 \text{ s}$$

The crossing time is 49 s.

(c) The current carries the canoe eastward (downstream) during the time it takes to cross the river. The downstream displacement is

$$\Delta \vec{d} = \vec{v}_{WS} \Delta t$$

$$= (3.2 \text{ m/s [E]})(49 \text{ s})$$

$$\Delta \vec{d} = 1.6 \times 10^2 \text{ m [E]}$$

The landing position is $2.2 \times 10^2 \text{ m [N]}$ and $1.6 \times 10^2 \text{ m [E]}$ of the starting position. Using the law of Pythagoras and trigonometry, the resultant displacement is $2.7 \times 10^2 \text{ m [36° E of N]}$.

(d) The velocity of the canoe relative to the water, $\vec{v}_{CW}$, which has a magnitude of 4.5 m/s, is the hypotenuse of the triangle in Figure 5. The resultant velocity $\vec{v}_{CS}$ must point directly north for the canoe to land directly north of the starting position.

Figure 5
The solution for part (d)

**LEARNING TIP**

Alternative Symbols
An alternative method of writing a relative velocity equation is to place the subscript for the observed object before the $\vec{v}$ and the subscript for the frame of reference after the $\vec{v}$. Using this method, the equation for our example of plane and air is $p\vec{v}_E = p\vec{v}_A + A\vec{v}_E$. 
The angle in the triangle is
\[
\phi = \sin^{-1} \left( \frac{|v_{\text{WS}}|}{|v_{\text{DW}}|} \right) = \sin^{-1} \left( \frac{3.2 \text{ m/s}}{4.5 \text{ m/s}} \right) = 45^\circ
\]

The required heading for the canoe is [45° W of N].

**SAMPLE problem 2**

The air speed of a small plane is 215 km/h. The wind is blowing at 57 km/h from the west. Determine the velocity of the plane relative to the ground if the pilot keeps the plane aimed in the direction [34° E of N].

**Solution**

We use the subscripts P for the plane, E for Earth or the ground, and A for the air.

\[
\begin{align*}
\vec{v}_{\text{PA}} &= 215 \text{ km/h [34° E of N]} \\
\vec{v}_{\text{AE}} &= 57 \text{ km/h [E]} \\
\vec{v}_{\text{PE}} &= ?
\end{align*}
\]

\[
\vec{v}_{\text{PE}} = \vec{v}_{\text{PA}} + \vec{v}_{\text{AE}}
\]

This vector addition is shown in Figure 6. We will solve this problem by applying the cosine and sine laws; however, we could also apply a vector scale diagram or components as described in Appendix A.

Using the cosine law:

\[
|\vec{v}_{\text{PE}}|^2 = |\vec{v}_{\text{PA}}|^2 + |\vec{v}_{\text{AE}}|^2 - 2|\vec{v}_{\text{PA}}||\vec{v}_{\text{AE}}| \cos \phi
\]

\[
|\vec{v}_{\text{PE}}| = 251 \text{ km/h}
\]

Using the sine law:

\[
\frac{\sin \theta_{\text{AE}}}{|\vec{v}_{\text{AE}}|} = \frac{\sin \phi}{|\vec{v}_{\text{PE}}|}
\]

\[
\sin \theta_{\text{AE}} = \frac{57 \text{ km/h (sin 124°)}}{251 \text{ km/h}}
\]

\[
\theta_{\text{AE}} = 11^\circ
\]

The direction of \(\vec{v}_{\text{PE}}\) is 34° + 11° = 45° E of N. Thus \(\vec{v}_{\text{PE}} = 251 \text{ km/h [45° E of N]}\).

Sometimes it is helpful to know that the velocity of object X relative to object Y has the same magnitude as the velocity of Y relative to X, but is opposite in direction: \(\vec{v}_{\text{XY}} = -\vec{v}_{\text{YX}}\). Consider, for example, a jogger J running past a person P sitting on a park bench. If \(\vec{v}_{\text{JP}} = 2.5 \text{ m/s [E]}\), then P is viewing J moving eastward at 2.5 m/s. To J, P appears to be moving at a velocity of 2.5 m/s [W]. Thus \(\vec{v}_{\text{PJ}} = -2.5 \text{ m/s [E]} = 2.5 \text{ m/s [W]}\). In the next Sample Problem, we will use this relationship for performing a vector subtraction.
A helicopter, flying where the average wind velocity is 38 km/h [25° N of E], needs to achieve a velocity of 91 km/h [17° W of N] relative to the ground to arrive at the destination on time, as shown in Figure 7. What is the necessary velocity relative to the air?

**Solution**

Using the subscripts H for the helicopter, G for the ground, and A for the air, we have the following relative velocities:

\[
\vec{v}_{HG} = 91 \text{ km/h [17° W of N]}
\]

\[
\vec{v}_{AG} = 38 \text{ km/h [25° N of E]}
\]

\[
\vec{v}_{HA} = ?
\]

\[
\vec{v}_{AG} = \vec{v}_{HA} + \vec{v}_{AG}
\]

We rearrange the equation to solve for the unknown:

\[
\vec{v}_{HA} = \vec{v}_{HG} - \vec{v}_{AG}
\]

\[
\vec{v}_{HA} = \vec{v}_{HG} + (-\vec{v}_{AG})
\]

where \(-\vec{v}_{AG}\) is 38 km/h [25° S of W].

Figure 8 shows this vector subtraction. By direct measurement on the scale diagram, we can see that the velocity of the helicopter relative to the air must be 94 km/h [41° W of N]. The same result can be obtained using components or the laws of sines and cosines.

**Answers**

2. (a) 3.9 m/s [fwd]  
   (b) 1.7 m/s [fwd]  
   (c) 3.0 m/s [21° right of fwp]  
3. 5.3 m/s [12° E of N]  
4. $7.2 \times 10^2$ km [30° S of W] from Winnipeg  

**Making Connections**

5. Airline pilots are often able to use the jet stream to minimize flight times. Find out more about the importance of the jet stream in aviation.
SUMMARY

Frames of Reference and Relative Velocity

- A frame of reference is a coordinate system relative to which motion can be observed.
- Relative velocity is the velocity of an object relative to a specific frame of reference. (A typical relative velocity equation is $\vec{v}_{PE} = \vec{v}_{PA} + \vec{v}_{AE}$, where P is the observed object and E is the observer or frame of reference.)

Section 1.5 Questions

Understanding Concepts

1. Two kayakers can move at the same speed in calm water. One begins kayaking straight across a river, while the other kayaks at an angle upstream in the same river to land straight across from the starting position. Assume the speed of the kayakers is greater than the speed of the river current. Which kayaker reaches the far side first? Explain why.

2. A helicopter travels with an airspeed of 55 m/s. The helicopter heads in the direction [35° N of W]. What is its velocity relative to the ground if the wind velocity is (a) 21 m/s [E] and (b) 21 m/s [22° W of N]?

3. A swimmer who achieves a speed of 0.75 m/s in still water swims directly across a river 72 m wide. The swimmer lands on the far shore at a position 54 m downstream from the starting point. (a) Determine the speed of the river current. (b) Determine the swimmer’s velocity relative to the shore. (c) Determine the direction the swimmer would have to aim to land directly across from the starting position.

4. A pilot is required to fly directly from London, UK, to Rome, Italy in 3.5 h. The displacement is $1.4 \times 10^3$ km [43° E of S]. A wind is blowing with a velocity of 75 km/h [E]. Determine the required velocity of the plane relative to the air.

Applying Inquiry Skills

5. A physics student on a train estimates the speed of falling raindrops on the train car’s window. Figure 9 shows the student’s method of estimating the angle with which the drops are moving along the window glass. (a) Assuming that the raindrops are falling straight downward relative to Earth’s frame of reference, and that the speed of the train is 64 km/h, determine the vertical speed of the drops. (b) Describe sources of error in carrying out this type of estimation.

Making Connections

6. You have made a video recording of a weather report, showing a reporter standing in the wind and rain of a hurricane. How could you analyze the video to estimate the wind speed? Assume that the wind is blowing horizontally, and that the vertical component of the velocity of the raindrops is the same as the vertical component for the raindrops in the previous question.